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Fractal properties of scattering singularities

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Abstract. We discuss properties of the set of scattering singularities in regions of irregular scattering. We show how a symbolic organisation of the set can be used to determine the fractal dimension and the scaling function. This yields information on the distribution of Lyapunov exponents of bounded orbits. The specific model studied is the motion of a particle in a plane, elastically reflected by three circular discs centred on the corners of an equilateral triangle.

1. Introduction

In a number of systems in celestial mechanics (Petit and Hénon 1986), molecular dynamics (Noid *et al* 1986), potential scattering (Eckhardt and Jung 1986) and hydrodynamics (Manakov and Shchur 1983, Eckhardt and Aref 1987), irregular scattering has been observed. The characteristic feature of irregular scattering is a clustering of initial conditions that lead to asymptotic trapping in the interaction region. Though still a set of measure zero and therefore in a sense irrelvant, these singularities are dressed with strong oscillations in the scattering angle and impact parameter and therefore make themselves felt in their immediate vicinity. In this paper, we continue our investigation of irregular scattering with a study of the fractal properties of the set of singularities.

Noid *et al* (1986) computed the fractal dimension of the graph of the outgoing scattering angle as a function of the impact parameter and found it to be close to two. Their study was made difficult by the large computational resources needed. Hénon (1987) has been able to derive analytically the fractal dimension for a piecewise linear scattering model. We believe, however, that piecewise linear models are exceptional in that they produce homogeneous fractals with trivial scaling functions. One of the purposes of the present contribution is to show that the set of singularities is a non-trivial fractal. To circumvent computational and genericity problems, we propose to study a billiard problem. As we will see, this allows us to approximate the set of scattering singularities quite accurately.

This model still has its shortcomings. Billiards are often used to illustrate completely chaotic dynamical systems (namely Bunimovich's stadium billiard or Sinai's billiard on a torus). The model we study below is of this type, if turned into a bound system by closing configuration space sufficiently far from the centre by either a circle or a

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rectangular box. In this sense, the results presented here are typical of scattering off 'completely chaotic objects'. It could be that for smooth potentials, clustering of islands near boundary circles resulting in an increased number of unstable orbits is important. This is outside the present model and needs to be investigated separately.

In § 2, we describe the model. We then study the fractal dimension and the scaling behaviour in § 3. In § 4 we characterise the set of scattering singularities as a set invariant under a topological mapping of phase space onto itself. This we use in § 5 where we discuss the relationship between instability exponents and fractal dimension. We conclude with a summary in § 6.

2. The model

The model we study here is a discontinuous variant of the smooth potentials of Eckhardt and Jung (1986) and Jung and Scholz (1987). It consists of three circular discs centred on the corners of an equilateral triangle (length of side $2\sqrt{3}$). Motion is confined to the exterior of the discs with the understanding that at collisions with the discs the particle is reflected elastically. Figure 1 shows the geometry and an orbit for illustration. For later use we willl label the discs clockwise, beginning in the upper right.

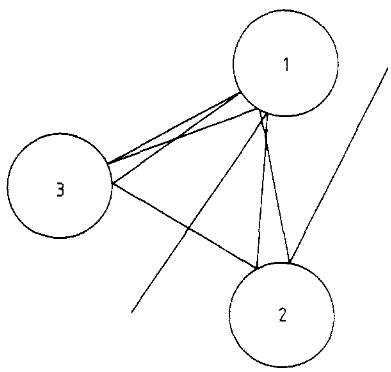

Figure 1. The scattering geometry with the three discs and a reference orbit with symbol sequence 1231312.

To describe the scattering behaviour of the model we introduce the ingoing and outgoing angles ϕ_{in} and ϕ_{out} and the corresponding impact parameters b_{in} and b_{out}, defined with respect to the centre of mass of the triangle. For (almost) any fixed initial angle there are three intervals in impact parameter in which the particle collides with one of the discs. For definiteness, we now specify an initial angle $\phi_{\text{in}} = 1.0$ rad and focus on collisions beginning with disc 1. The corresponding interval will be called C_1. The salient feature of irregular scattering is the existence of trajectories that stay within the 'interaction' region for arbitrarily long times. For the billiard model this means an infinite number of collisions. We therefore begin by asking in what intervals does the particle collide at least twice? Since the scattering data after one collision are smooth functions of the initial data, it suffices to determine the impact parameter for trajectories that are tangential to the other discs. This gives two intervals, labelled C_{12} and C_{13}, in which the particle collides first with disc 1 and then with disc 2, or first with disc 1 and then with disc 3, respectively. Similar reasoning allows us to determine intervals C_{121}, C_{123}, C_{131} and C_{132} in which the particle collides at least three times. The general picture should now be clear: at each 'generation' n of collision sequences, there are 2^{n-1} disjoint intervals in impact parameter leading to at least n collisions. They can be labelled uniquely by the symbols $(i_1, i_2, \ldots, i_n)$ of the collision sequence of a typical orbit in the interval: i_1 stands for the number of the disc for the first collision, i_2 for that of the second, and so forth. Of course, two consecutive symbols must not be identical. Other than that, any sequence can be realised.

The set of singularities is then given by

$$C = \lim_{n\to\infty} \bigcap_{i\in S_n} C_i \tag{1}$$

where S_n denotes the set of all allowed symbol sequences of length n and $\boldsymbol{i}$ is a vector of symbols. The construction of this set is very similar to that of the usual ternary Cantor set.

3. Fractal dimension and scaling function

The simplest measure of the fractal properties of this set is its fractal dimension d_F, defined by the asymptotic scaling behaviour of the number $N(\varepsilon)$ of intervals of size ε needed to cover the set:

$$N(\varepsilon) \approx \varepsilon^{d_F} \qquad \text{as } \varepsilon \to 0. \tag{2}$$

In the case at hand, a slight modification suggests itself. It is easier to determine the total area A_n of all intervals of generation n and its scaling behaviour. Figure 2 shows data for up to ten generations and a fit $A_n \approx e^{-\gamma n}$. As an example, for a radius $r = 0.7$ one finds $\gamma = 1.42$. If ε_n denotes the size of a typical interval for the covering at generation n, then

$$N(\varepsilon_n) \approx \varepsilon_n^{d_F} \approx 2^{n-1} \tag{3}$$

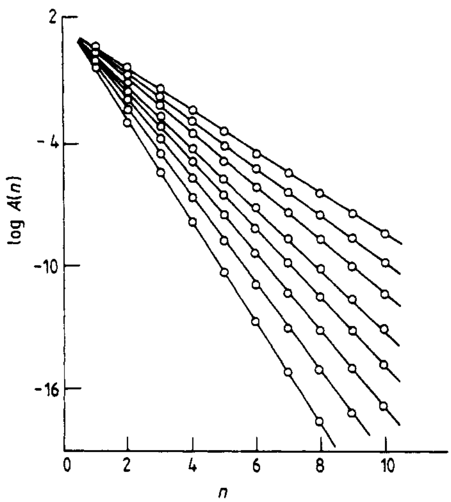

Figure 2. The total area of the intervals at generation n against the number of the generation. The different curves correspond to the following radii of discs (from top to bottom): 1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4 and 0.3. The different symbols at radius 0.7 correspond to different initial angles.

by the definition of N and the observation on the number of intervals made earlier. This may now be solved for ε_n. Given $N(\varepsilon_n)$, the area of the covering is

$$A(\varepsilon_n) \approx \varepsilon_n N(\varepsilon_n) \approx \varepsilon_n^{1+d_F} \approx e^{-\gamma n}. \tag{4}$$

Solving for d_F, one finds

$$d_F = \frac{\log 2}{\log 2 + \gamma}. \tag{5}$$

For our example with radius $r = 0.7$ and the data from figure 2, one finds $d_F \approx 0.324$. The data also suggest that d_F is insensitive to the initial angle. We will return to this point below.

For our further analysis of the set of singularities, it will be convenient to replace the ternary symbol sequences with restrictions by binary symbol sequences without restrictions. This is also suggested by the binary splitting of the intervals in going from one generation to the next. We associate with any (clockwise) collision sequence $1 \to 2$, $2 \to 3$ or $3 \to 1$ the symbol L and with any (counter-clockwise) collision sequence $1 \to 3$, $2 \to 1$ or $3 \to 2$ the symbol R. Given that the first collision occurs with disc 1, this relabelling uniquely specifies a collision sequence. As an example, the sequence from figure 1, 1231312, now becomes *RRRLRR*.

The fractal dimension as discussed before is a global, averaged quantity of the fractal set. Much more detailed information may be obtained from the thermodynamic formalism for strange sets, in particular the scaling function $\sigma(t)$ (see Benzi *et al* (1984), Halsey *et al* (1986) and Feigenbaum *et al* (1986) for details). Let ε_i denote the binary symbols as defined in the last paragraph and let $l(\varepsilon_n, \varepsilon_{n-1}, \ldots, \varepsilon_1)$ be the length of the interval in which a trajectory beginning with $\varepsilon_1, \ldots, \varepsilon_n$ can be found. Then the scaling function $\sigma(\varepsilon_{n+1}, \varepsilon_n, \ldots, \varepsilon_1)$ is defined as

$$\sigma(\varepsilon_{n+1}, \ldots, \varepsilon_1) = \frac{l(\varepsilon_{n+1}, \varepsilon_n, \ldots, \varepsilon_1)}{l(\varepsilon_n, \ldots, \varepsilon_1)} \tag{6}$$

i.e. as the ratio of the length of the daughter interval in generation $n+1$ to that of the mother interval in generation n. Trivial fractal sets, like the standard $\frac{1}{3}$ Cantor set have trivial scaling functions depending only on where one is going, $\sigma = \sigma(\varepsilon_{n+1})$. Less trivial sets have memory, $\sigma = \sigma(\varepsilon_{n+1}, \varepsilon_n, \varepsilon_{n-1}, \ldots)$, all the way up to sets like the attractor at the period 2^∞ accumulation point in period doubling, which has an infinite number of scales (Aurell 1987). The more complicated the organisation of the fractal set, the longer the memory in σ. For the set of scattering singularities, we find the scaling function $\sigma(t)$ shown in figure 3, where

$$t = 2^{-n} \sum_{k=1}^{n} \varepsilon_k 2^{k-1} \tag{7}$$

and where the numerical values of 0 and 1 have been assigned to the symbols L and R, respectively. (Note that ε_n carries the largest weight.) The figure shows that in order to get a meaningful approximation to the scaling function, one needs to take into account at least a one-step memory, i.e. two symbols. Thus, $\sigma = \sigma(\varepsilon_n, \varepsilon_{n-1})$, where roughly

$$\sigma(L, L) \approx \sigma(R, R) < \sigma(L, R) \approx \sigma(R, L). \tag{8}$$

This is in agreement with intuition, which tells us that it is easier for the particle to be reflected back to where it came from (the elements $\sigma(L, R)$ and $\sigma(R, L)$) than it

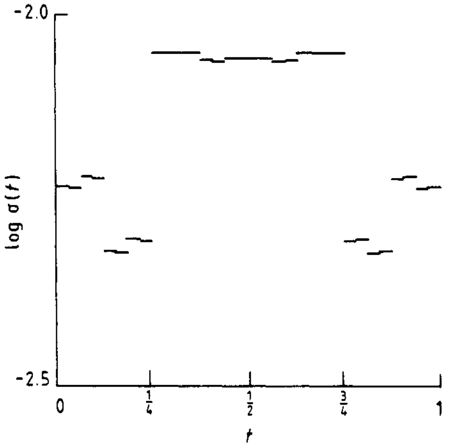

Figure 3. The scaling functions as determined for the fractal set at radius 0.7. The results from approximations using 6, 7, 8, 9 and 10 generations have been superimposed. Deviations are below the resolution of the linewidth.

is for it to be deflected to the third disc. The numerical calculations also show that $\sigma(t)$ settles to its asymptotic behaviour after about five generations, and that the set as determined from generations 2 and 3 satisfies relationship (8) quite well.

4. The topological origin of the fractal

We now turn to a global discussion of the origin of the Cantor set structure. In figure 4 we show the phase space near disc 2. The angle θ is the polar angle of the point of collision on the circumference of the disc and ϕ is the ingoing direction of the velocity. All points within region A will be reflected towards disc 1. The width of this region is given essentially by the opening angle of the second disc as seen from the point

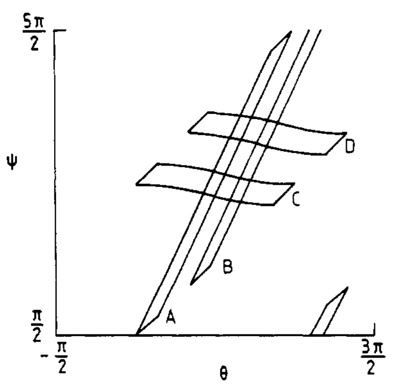

Figure 4. Phase space near one disc. The angle ϕ is the angle of the ingoing velocity and the angle θ determines the point of impact along the boundary of the disc. Regions A and B contain all the points mapped onto discs 1 and 3, respectively, upon reflection. Regions C and D mark those parts of phase space that can be reached from other spheres. This construction is used in the text to explain the fractal nature of the scattering singularities.

determined by θ, unless the second disc has not risen above the horizon; the latter case leads to the linear boundary segments. Similarly, all points in region B will be reflected towards disc 3. The domain is derived from A, shifted by $\pi/3$ in both coordinates. In the binary notation introduced before, region A corresponds to symbol L and region B to symbol R.

Because of the threefold symmetry of the disc arrangment, phase space diagrams for the other two discs are simply shifted images of the one shown here: a shift by π results in the diagram for discs 1 and one by $-2\pi/3$ results in the diagram for disc 3.

We also need the image of all points deflected to disc 2 from discs 1 or 3. These are shown as regions C and D. Again, it is sufficient to determine one and to obtain the other by translation. The important feature is that all four regions overlap. As we will now discuss, this leads to the Cantor set structure. Symmetry allows us to confine ourselves to one disc and one phase space diagram. Again the binary coding will prove the most convenient. The full scattering trajectory can be retrieved, if the shifts as determined from the symbol sequence are taken into account.

So we start from two regions A and B containing initial conditions leading to collisions with other discs. The images of these sets are $C = f(A)$ and $D = f(B)$, where the function f stands for the action of one scattering process. We now associate symbolic sequences with these sets. Initial conditions in A and $f(A)$ lead to counter-clockwise motion, so they correspond to the symbol L, whereas B and $f(B)$, leading to clockwise motion, correspond to the symbol R. Any point in the intersection of, say, $A \cap f(B)$ will therefore have a symbol sequence LR; similarly for the others as shown in figure 4. The second iterate of the mapping f is only defined on the intersection regions. They are smaller than the original domains, so their image is smaller also. This contraction of the domain of definition with an increasing number of required collisions is familar from the horseshoe construction (Moser 1973). There one also has a set and a mapping so that the image and the domain of the mapping intersect. An infinite number of iterations is only possible in the intersection of the set and all its iterates, typically yielding a Cantor set.

The way our scattering initial conditions are set up, we study a cross section of the resulting Cantor set. Invariance of the scaling behaviour, the scaling function and the $f(\alpha)$ curves (see below) suggests that all cross sections look alike.

5. Instability of orbits and fractal dimension

The intersection of all forward and backward iterates of the mapping contains all bound orbits. Since all (binary) symbol sequences can be realised, these bound orbits need not be periodic. Linearisation of the motion perpendicular to the orbits allows one to determine the Lyapunov characteristic exponents. In our case, they will always be positive, since the orbits are unstable. Suppose we start with a small square in the phase space near one disc, containing bound orbits; the square should be sufficiently small to allow at least N iterations for all points in it. Then at each step, the expansion rate of the square will roughly be given by an average over the Lyapunov exponents of the bound orbits contained in it. But this rate is also an estimate of how much of the iterated square will intersect the original square. The conclusion to be drawn from this line of reasoning is that the width of the intervals allowing N collisions is related to the stability properties of bound orbits. As the number of collisions increases, the collision orbit follows bound orbits for ever longer times, so in the limit the distribution of scales in the fractal reflects the distribution of Lyapunov exponents in the bound motion.

All of them are unstable. The mapping is area contracting, so none can have an island of stability to go with it. Locally, i.e. near a given bounded orbit, the contraction rate is given by the instability of the orbit. Therefore it seems reasonable to suggest that locally the fractal dimension of the set is determined by the instability of the orbits.

To check this idea locally is very difficult since, given a scattering trajectory and its finite symbol sequence, it is not at all clear to which periodic or non-periodic bounded orbit it belongs. Therefore, the following test was performed. For any interval, we computed the orbit in its centre, and determined the sensitivity of the outgoing angle and impact parameter on the initial impact parameter, $\Delta\phi_{out} = \lambda_\phi \Delta b_{in}$ and $\Delta b_{out} = \lambda_b \Delta b_{in}$. The products $\lambda_\phi l$ and $\lambda_b l$ formed out of the sensitivities of the orbit and the length of the intervals was constant within any generation and the same constant for different generations for systems with the same radius of the discs, independent of the initial angle. The variations of these numbers were about 0.01%, compared to variations in the size of the interval of up to 50%. We take this to be very good numerical evidence for the proposed connection between the size of the intervals and the fractal structure of the set.

Taking this for granted, we can argue in the reverse. If we know details of the fractal organisation of scattering singularities, we get information on the stability properties of bounded orbits. A convenient way to present this information is the function $f(\alpha)$ introduced by Halsey *et al* (1986). The Lyapunov exponents are related to α and $f(\alpha)$ is the fractal dimension of the set of bounded periodic orbits with that Lyapunov exponent.

As discussed in the introduction of Halsey *et al* (1986), the variable α is determined from the generalised entropies

$$D_q = \lim_{l\to 0}\left(\frac{1}{q-1}\,\frac{\log\chi(q)}{\log l}\right) \tag{9}$$

where

$$\chi(q) = \sum_i p_i^q \tag{10}$$

via

$$\alpha(q) = \frac{\mathrm{d}}{\mathrm{d}q}[(q-1)D_q]. \tag{11}$$

Assuming that at generation n, all periodic orbits carry the same weight, the probabilities become equal to $p_i = 2^{-(n-1)}$, so that $\chi(q) = 2^{-(q-1)(n-1)}$. Let λ denote the Lyapunov exponent on which we wish to concentrate. Then the length l of the intervals decreases like $l \approx \mathrm{e}^{-\lambda(n-1)}$, so that

$$D_q = \log 2/\log\lambda. \tag{12}$$

Thus,

$$\alpha(q) = \log 2/\log\lambda. \tag{13}$$

This is the required relationship between the Lyapunov exponent and the parameter α. Figure 5 now shows an example of $f(\alpha)$ for a radius $r = 0.7$. Good convergence is achieved after four generations. Note the finite range of α values over which $f(\alpha)$ does not vanish: it indicates that all Lyapunov exponents fall in a finite interval.

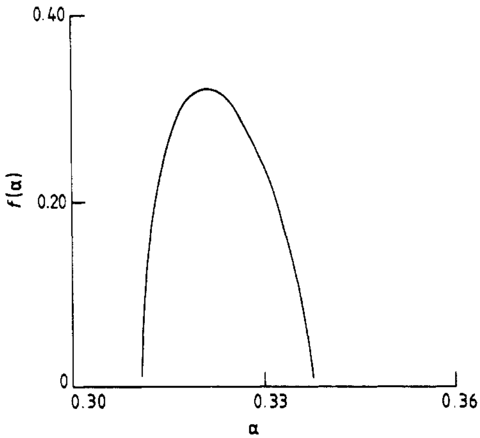

Figure 5. The singularity function $f(\alpha)$ for the fractal set at radius 0.7.

6. Conclusions

For the billiard model studied here, we have determined the fractal dimension and the scaling function for the set of scattering singularities. Good convergence was found with only 10 generations in the symbolic organisation of the set. The scaling function may be approximated qualitatively with only four scales, and 16 are enough to reproduce it quantitatively. Clearly, the scattering singularities lie in a multifractal set. The fractal dimension was found to be linked to the stability properties of the bound orbits asymptotically approached by the scattering trajectory: the more unstable these orbits, the smaller the fractal dimensions. Different orbits can have different Lyapunov exponents. The range of Lyapunov exponents can be described using the singularity function $f(\alpha)$.

These are the main results of the present work. We conclude with a conjecture on potential scattering. There, it is conceivable that some bounded orbits have just become unstable, i.e. have very small Lyapunov exponents. This corresponds to large values of α. The $f(\alpha)$ curves then extend to very large values of α and perhaps are rather flat near the maximum, making it very difficult to determine the fractal dimension (which is given by the maximum). This could explain some of the numerical difficulties with smooth potentials.

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